Scaling and universality in the micro-structure of urban space

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Abstract

We present a broad, phenomenological picture of the distribution of the length of urban linear segments, *l*, derived from maps of 36 cities in 14 different countries. By scaling the Zipf plot of *l*, we obtain two master curves for a sample of cities, which are not a function of city size. We show that a third class of cities is not easily classifiable into these two universality classes. The cumulative distribution of *l* displays power-law tails with two distinct exponents, $\alpha_S = 2$ and $\alpha_R = 3$. We suggest a link between our observations and the possibility of observing and modelling urban growth using Levy processes.

Keywords
Fractals, urban planning, scaling laws, universality

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1. Introduction

The morphology of urban settlements and its dynamics has captured the interest of researchers [1, 2, 3, 4, 5, 6, 7, 8, 9] as it may shed light on Zipf's law for cities [4, 5, 10, 11], challenge theoretical frameworks for cluster dynamics or improve predictions of future urban growth [2, 6, 7].

The search for a unified theory of urban morphology has focused on the premise that cities can be conceptualized at several scales as fractals. At the regional scale, rank-order plots of city size follow a fractal distribution [1] and population scales with city area as a power-law [12]. More recently, it has been observed that the area distribution of satellite cities, towns and villages around large urban centres also obeys a power-law with exponent ≈ 2 [2, 6]. At the scale of transportation networks, railway networks appear to have a fractal structure [13]. At the scale of the neighbourhood, it has been suggested that urban space resembles a Sierpinsky gasket [1, 12]. These scales are inter-related as summed up in [1: 241] (author's translation from French): *'Polycentric growth, which is connected to the non-homogeneous distribution of pre-urban cores and the birth* *of a hierarchy of sub-centres, influences the morphology of the transport network, which plays in itself an important role in axial growth and therefore for the future spatial development of the urbanised area'.*

The fractal dimensions of US cities and international cities have values ranging from 1.2778 (Omaha, [14]) to 1.93 (Beijing, [1]), where the fractal dimension of large cities tends to cluster around the latter value [1, 12, 15, 14]. Studies of urban growth of London between 1820 and 1962 show that fractal dimensions for this period vary from 1.322 to 1.791 [12]. The fractal dimensions for the growth of Berlin in 1875, 1920 and 1945 are 1.43, 1.54 and 1.69, respectively [1]. The fractal dimension of urban aggregates is a global measure of areal coverage, but detailed measures of spatial distribution are clearly needed to complement adequately the description of the morphology of an urban area [7]. Further, current approaches to data collection and modelling identify cities as fractal only on the urban periphery of the giant urban cluster that grows around the city core (or central business district), as clusters become compact at distances close to the centre of the city [12, 6]. Although remote sensing techniques are promising in extracting urban morphology with greater detail, available studies have, to our knowledge, been limited to individually selected, medium scale cities (see e.g. [16]).

Hillier and Hanson [17] suggest an underlying structure to urban open space that is determined by the complexity of buildings which bound the space [18]. Urban space available for pedestrian movement, excluding by definition physical obstacles, is relatively linear. When people walk through this open space, they perceive it locally as a 'vista' which can be represented approximately as a line. The global set of vistas, the so-called *axial map, is* defined as the least number of longest straight lines. An axial map can be derived by drawing the longest possible straight line on a city map, then the next longest line, so-called axial line, until the open space is crossed and 'all axial lines that can be linked to other axial lines without repetition are linked' [17, 19]. Figure 3 shows several axial maps.

Axial maps may be relevant to researchers interested in urban morphology. They provide a simplified signature of the growth process, since the analysis is restricted to the linearity of open space. Indeed, one could hope to extract the spatio-temporal dynamics of axial map growth by analysing a sequence of aerial photographs of the urban periphery over a period of growth. Conversely, one could hope to model urban growth as the trajectory of N walkers on a plane, the step of the walk being axial line length.

Here we show that we can rescale axial line length and rank to obtain two distinct rank-order curves that provide a classification for several cities independently of city size. We also show that there is a class of cities that do not obey this classification. The collapse of curves suggests that spatial fluctuations in the length of urban linear structures, differing in size and location, are governed by similar statistical rules and supports the hypothesis that the linear dimension of large scale structures in cities reflects generic properties of city growth [20].

1.1 Can physicists contribute to urban science?[1]

Statistical physics deals with systems comprising a very large number of interacting subunits, for which predicting the exact behaviour of the individual subunit would be impossible. Hence, one is limited to making statistical predictions regarding the collective behaviour of the subunits [21]. Recently, it has come to be appreciated that many such systems which consist of a large number of interacting subunits obey universal laws that are independent of the microscopic details. The finding, in physical systems, of universal properties that do not depend on the specific form of the interactions gives rise to the intriguing hypothesis that universal laws or results may also be present in economic and social systems [22, 23][2].

If there is one message that emerges clearly from statistical physics, it is that sometimes *the details do not matter*. That, in a nutshell, is what is meant by universality. This is a way of saying that *collective* behaviour tends to be robust, and shared by many apparently different systems [24].

1.1.1 What is scaling? and universality?

Scaling may be expressed as a relatively compact statement:

$$p(\lambda l) = \lambda p(l)$$

A scale invariant system has the same statistical properties and hence "looks almost the same" at many different scales of observation. The actual object is different, but since its *statistical* properties are the same, one cannot readily distinguish the original complex object from a magnification of a part of it [25].

For a vivid analogy, recall the infamous Tacoma Narrows Bridge that once connected mainland Washington with the Olympic peninsula[3]. One day it suddenly collapsed after developing a remarkably "ordered" sway in response to a strong wind. Students learn the explanation for this catastrophe: the bridge, like most objects, has a small number of characteristic vibration frequencies, and one day the wind was exactly the strength required to excite one of them. The bridge responded by vibrating at this characteristic frequency so strongly that it fractured the supports holding it together. The cure for this "diseased bridge" was a design that is capable of responding to many different vibration scales in an approximately equal fashion, instead of responding to one frequency excessively.

What about universality, the notion in statistical physics that many laws seem to be remarkably independent of details?

It was found empirically that one could form an analog of the Mendeleev table if one partitions all critical systems into "universality classes". It was found that quite disparate systems behave in a remarkably similar fashion near their respective critical points -simply because near their critical points what matters most is not the details of the microscopic interactions but rather the nature of the "paths along which order is propagated" [26]. As we shall see, the fact that data for each city collapse onto a scaling function supports the scaling hypothesis, while the fact that the scaling function is the *same* for several different cities is truly remarkable. Two cities with the same values of the scaling exponent are said to belong to the same universality class. Thus the fact that the exponents are the same for e.g. London and Athens, but different from the exponents for e.g. Tokyo and Bangkok, implies that the former belong to a distinct universality class from the latter.

Newcomers to the field of scaling invariance often ask why a power-law does not extend 'forever' as it would for a mathematical power-law of the form $f(x) = x^{-\alpha}$. This legitimate concern is put to rest by reflecting on the fact that power-laws for natural phenomena are not equalities, but rather are asymptotic relations of the form $f(x) \sim x^{-\alpha}$ Here the tilde denotes *asymptotic equality.* This means that *f(x)* becomes increasingly like a power law as $x \to \infty$. For a discussion on scaling and universality, the reader should refer to [23, 25].

1.1.2 What are power-law distributions?

Frequency or probability distribution functions (pdf) that decay as a power-law of their argument

$$p(x)dx = p_0 x^{-(1+\alpha)} dx$$

have acquired a special status in the last decade and are sometimes called "fractal". A power-law distribution characterizes the absence of a characteristic size: *independently* of the value of x, the number of realizations larger than λx is $\lambda^{-\alpha}$ times the number of realizations larger than x. In contrast, an exponential for instance does not enjoy this self-similarity[4], as the existence of a characteristic scale destroys this continuous scale invariance property [27]. In words, a power-law pdf is such that there is the same proportion of smaller and larger events, whatever the size one is looking at within the power-law range.

Power-law pdfs have the characteristic that the sample mean does not approach a limiting value as more data is collected (the averaged measures will either increase or decrease with the amount of data analysed). There is no single value that we can identify as the "right" value for the average. Therefore, the population mean does not exist. For an insightful discussion on power-law pdfs see e.g. [28].

Researchers may care passionately if there are analogies between physics systems they understand (like critical point phenomena) and urban systems they do not understand. But why should anyone else care? One reason is that scientific understanding of earthquakes moved ahead after it was recognized that extremely large events -previously regarded as statistical outliers requiring for their interpretation a theory quite distinct from the theories that explain everyday shocks- in fact possess the identical statistical properties as everyday events; e.g., all earthquakes fall on the same straight line on an appropriate log-log plot. Since, as we shall see, urban phenomena possess the analogous property, the challenge is to develop a coherent understanding of urban morphology that incorporates not only the space where we navigate daily, but also the extremely rare 'morphological earthquakes'.

Finally, a current interesting hypothesis is that possibly one reason that diverse systems in such fields has physics, biology, and ecology have quantitative features in common may relate to the fact that the complex interactions characterizing these systems could be mapped onto some geometric system, so that scaling and universality features of other complex systems may ultimately be understood in terms of the connectivity of geometrical objects [25].

2. Structure of urban space

Let $\mathbf{l}_i = \{\mathbf{l}_{i,j}\}$, $j = 1, \cdots, N_i$ be the N_i axial lines associated with city i. Each axial line, $\mathbf{l}_{i,j}$ $j = 1, \cdots, N_i$ is defined by the coordinates of its extremities

$$\mathbf{l}_{i,j} = \{(x_{(i,j),1}, y_{(i,j),1}), (x_{(i,j),2}, y_{(i,j),2})\}$$

The axial map of city i, C_i is thus a set of N_i points on a fourth dimensional space, $C_i = \{(\rho_{i,j}, \theta_{i,j}, l_{i,j}, \varphi_{i,j})\}$ where (ρ_i, θ_i) are the polar coordinates of the axial line geometric centre, $\mathbf{s}_{i,j} = \left(\frac{x_{(i,j),1}+x_{(i,j),2}}{2}, \frac{y_{(i,j),1}+y_{(i,j),2}}{2}\right)$ and $\left(\pm\frac{l_{ij}}{2}, \varphi_{i,j}\right)$ are the polar coordinates of the axial line's extremities on its geo metric centre reference system, $\mathbf{d}_{i,j} = \pm\left(\frac{|x_{(i,j),1}-x_{(i,j),2}|}{2}, \frac{|y_{(i,j),1}-y_{(i,j),2}|}{2}\right)$

Coordinates ρ and θ encode the geographic location of axial lines. The unconditional distribution of φ is multimodal for rather general families of urban settlements. This occurs, for example, when land is partitioned in clusters of randomly oriented orthogonal grids. Nevertheless, the unconditional distribution of l is unimodal and skewed to the right (see Figure 1), and, thus, a good candidate for inspection of intermittency in urban space. We fit the data to a stretched exponential distribution [29: 153-154] in Figure 1 (a), but verify that the fit is unsuitable to describe the large events.

Figure 1: Data are shown for the city of Tokyo. (a) Rank-order plot of line length (circle points) together with a fit of the data to a stretched exponential pdf (solid line). (b) Unconditional probability density of line length.

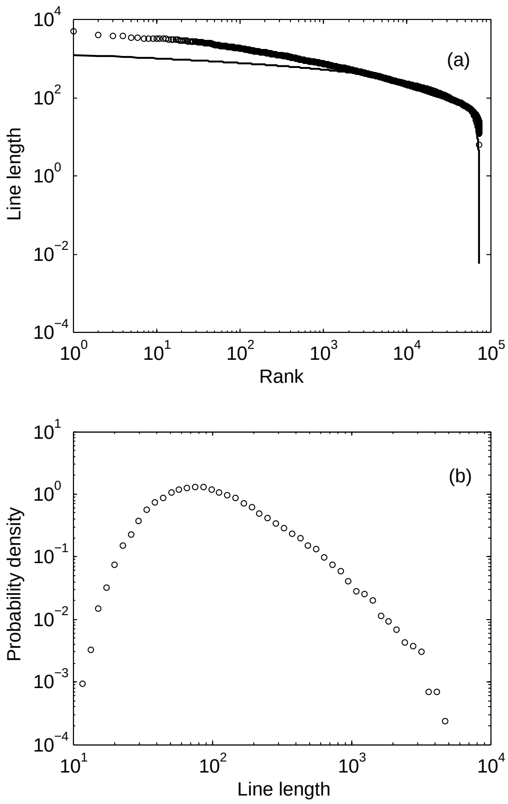

Table 1: Geographical location and number of lines of the cities analysed

Country	City	Number of lines
Japan	Tokyo	73753
U.S.A.	Chicago	30571
Chile	Santiago	26821
Thailand	Bangkok	24223
Greece	Athens	23329
Turkey	Istanbul	21798
U.S.A.	Seattle	20213
U.K.	London	15969
U.S.A.	Baltimore	11636
Netherlands	Amsterdam	9619
U.K.	Bristol	7028
U.S.A.	Las Vegas	6909
Iran	Shiraz	6258
Cyprus	Nicosia	6023
Netherlands	Eindhoven	5782
U.K.	Milton Keynes	5581
Spain	Barcelona	5575
U.K.	Wolverhampton	5423
India	Ahmenabad	4876
U.S.A.	New Orleans	4846
Iran	Kerman	4372
U.K.	Nottingham	4365
U.K.	Manchester	4308
U.S.A.	Pensacola	4296
Iran	Hamadan	3855
Iran	Qazvin	3723
Netherlands	The Hague	3350
U.K.	Norwich	2119
U.S.A.	Denver	2092
Iran	Kermanshah	1870
U.K.	York	1773
Iran	Semnan	1770
Bangladesh	Dhaka	1566
Hong Kong	Hong Kong	916
U.K.	Hereford	854
U.K.	Winchester	616

3. Inverse square and cubic laws for the distribution of line length

We analyse the unconditional probability distribution of (axial) line length of 36 cities in 14 different countries (see Table 1). In our analysis we use the rank-order technique [29]. To interpret the apparently unsystematic data in Figure 2(a) effectively, it is instructive to scale the data. Since the rank ranges between 1 and $\max(rank_i)$, we define a scaled relative rank for city i, $\widehat{r_{ij}} = rank_{ij}/\max(rank_j)$. Similarly, for the ordinate, it is useful to define a scaled line length by $\widehat{l_{ij}} = l_{ij}/\langle l_j \rangle$ [30] .

As shown in Figure 2(d), there is relatively good collapse of the data sets onto two master curves for 28 of the 36 cities under study (the cities plot in red and blue). The other 8 cities do not collapse clearly onto a single curve (see Figure 2(b)). Figure 2(c) is a plot of the exponents from a least-squares fit to the data of Figure 2(b) for $\log \hat{l}_{ij} > 0.2$, where the data are visually the most linear. The fits on the rank order plot lead to straight lines with slope $-1/\alpha$, which suggest that the line length probability density may have a power-law tail, $P(l) \sim l^{-1-\alpha}$ with exponents close to $\alpha_g \simeq 2$ (cities in blue) or $\alpha_g \simeq 3$ (cities in red). The inverse square and cubic laws have diverging higher moments (larger than 2 and 3, respectively) and are not stable distributions.

Figure 3 is a plot of several axial maps, where we only plot lines with $\log \hat{l}_{ij} > 0.2$ (the range of data used in the least-squares fit of Figure 2). We suggest that urban growth can be regarded as a process where axial lines are added to the urban periphery (this could, in principle, be monitored through remote sensing techniques for cities undergoing rapid urbanization) and

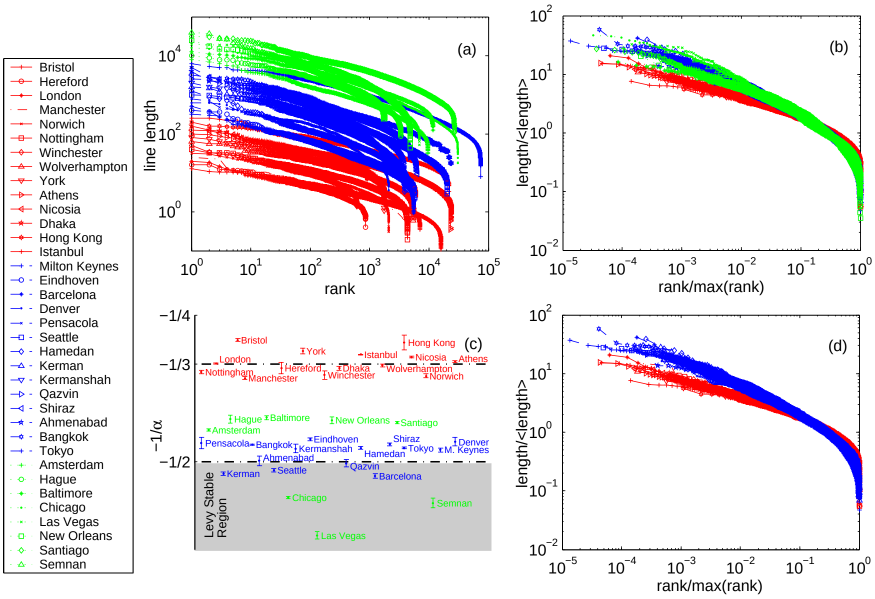

Figure 2: (a) Rank-order plot of line length versus rank. Consecutive curves have been vertically shifted for clarity. (b) Data in (a) in scaled units. (c) Exponents determined from least squares fits to the log-log data in (b) for $10^{0.2} < y$. Error bars are 95% confidence bounds. (d) Data in (b) excluding cities in green. Cities are coloured according to their ordinate in (c), and we have coloured a group of cities in green as they deviate considerably from the two universality classes in (d).

Figure 3: Axial maps of a sample from Figure 2 of cities coloured in blue (Tokyo and Bangkok), red (Athens) and green Las Vegas).

modelled by N walkers, which jump along the corresponding N axial lines, extending the city. More needs to be known on the distribution of the walkers' waiting time to correctly model the dynamics of urban growth. Nevertheless, the walkers would generate a non-stable process, as the exponents α_B and α_R are, apparently, outside of the Levy stable region.

4. Discussion

We have found that the length of urban open space structures displays universal features, largely independent of city size, and is self-similar across morphologically relevant ranges of scales (2 orders of magnitude) with exponents $\alpha_B \simeq 2$ (cities in blue) and $\alpha_R \simeq 3$ (cities in red). Our results are unexpected as two universality classes appear for a wide range of cities. The power-law tails of the pdfs support the hypothesis that urban space has a fractal structure [1, 12], but the parallelism to a Sierpinski gasket [1, 12] may be too simple for an accurate description.

Our findings show that it is important to model in detail the open space geometry of urban aggregates. They also support the hypothesis that it is more useful to model urban morphology as random rather than as the outcome of rational decisions, as previously suggested [12, 2, 6].

We interpret the difference in the exponents for the two groups of cities with 'macro' distinctions in urban planning policies. Cities with exponents $\alpha_B \simeq 2$ (cities in blue) display open space alignments which cross the whole structure whilst cities with $\alpha_R \simeq 3$ (cities in red) tend not to. We believe this can be explained by the dominance of global planning for the former as opposed to local planning for the latter.

Researchers have found exponents $\alpha \sim 3$ when studying the distribution of normalized returns in financial markets, both for individual companies [31, 32] and for market indexes [33]. Our results for the class of cities plot in red on Figure 2 is reminiscent of these studies. We believe that parallels between urban growth and finance may not be too far fetched, as both processes seem to be largely dominated by geometric phenomena [8, 34]. Indeed, there may be similarities between the dynamics of price fluctuations and urban growth, and we propose that axial lines may be seen as the urban equivalent of economic returns.

As more data becomes available through remote sensing, quantitative analyses should provide an improved view of the spatio-temporal dynamics of urban growth, particularly in squatter settlements, where time-scales for growth are much shorter than in conventional cities and one could hope to model growth against observed data. Further studies are still required, but it seems that the impact of local controls on growth (e.g. the green belt policy for London) is, at most, spatially localized. Indeed, at a 'macro' level, cities display a surprising degree of universality.

Notes

[1]This section is based on extracts from the references in the text.

[2]An often-expressed concern regarding the application of physics methods to the social sciences is that physics laws are said to apply to systems with a very large number of subunits (of order of $10^{2}0$ while social systems comprise a much smaller number of elements. However, the "thermodynamic limit" is reached in practice for rather small systems. For example, in early computer simulations of gases or liquids reasonable results are obtained for systems with 20 - 30 atoms.

[3]The bridge collapsed on November 7, 1940 at approximately 11:00 a.m. and had been open to traffic for only a few months. The reader is invited to view historical film footage which shows in 250 frames (10 s!) the maximum torsional motion shortly before failure of this immense structure [http://cee.carleton.ca/Exhibits/Tacoma-Narrows/].

[4]An object is said to be self-similar if it looks "roughly" the same on any scale.

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