On the Characterisation of Axial Maps

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0 Abstract

We propose a characterization of axial maps by means of an extension of a local property which has been proved to describe the cell decomposition of the plane induced by line arrangements. This gives a discrete model which is much closer from axial maps than the usual representation by graphs.

How is the urban grid to be represented as a set of discrete elements to make configurational analysis possible? ... What seems to be needed is a representation less general than the topology of the node graph, yet less precise than the geometry of the form itself. Hillier et al (1993) pp 33-34.

1 Introduction

Space syntax is a descriptive technique of spatial and configurational analysis developed at the Unity of Architectural Studies, University College London (Hillier et al (1983), Hillier and Hanson (1984), Hillier et al (1987a), Hillier et al (1987b)). It aims to describe space by means of a non-arbitrary and reproducible representation. The understanding of morphological structures, the quantification and modelling of configurational properties and the comparison between different spatial systems constitute its main purposes.

This approach considers space in terms of abstract properties of topological nature rather than in terms of geometric measures. It describes spatial layouts regarding the pattern of connections between spaces and quantifies the extend to which each space is directly connected to other spaces.

Figure 1. A segment selected in Castelo, Lisbon

Axial maps are graphical configurations introduced by Hillier and Hanson (1984) to describe morphological properties of urban forms. These objects allow a quantitative analysis of spatial layouts. The axial map is a planar connected configuration consisting of the fewest longest straight lines covering all urban public spaces. These lines correspond to the image of physical and visual continuity tested by people who are static or in movement in the system. Figure 1 is a segment selected in Castelo, a medieval part of Lisbon. This area shows an irregular urban layout characteristic of traditional towns which have grown organically. It reproduces the informal arrangement of vernacular site made up of blocks of outward-facing buildings with different shapes, narrow streets, paths and small squares.

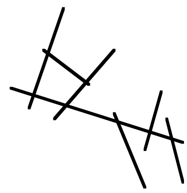


Figure 2. The corresponding axial map.

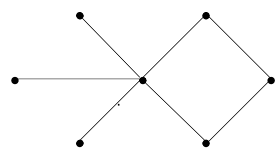


Figure 3. Axial graph of the map in figure 2.

Figure 2 shows the corresponding axial map. The axial map provides information about the pattern of connections between spaces (i.e., the way the lines are distributed on the plane) and the connections of each space to all other spaces (i.e., the intersections of each line to all other lines). No information in terms of areas and distances is given. To make an axial map suitable for computation it must somehow be converted into a set of discrete elements. The usual procedure is to transform the map into a graph: the so-called axial graph (Hillier and Hanson (1984)). The axial graph of an axial map is a graph in which vertices correspond to lines. Two vertices are adjacent if and only if the corresponding lines of the axial map intersect. The axial graph of the map of Figure 2 is represented in Figure 3.

Since graphs can be represented by 0-1 square matrices expressing the adjacency between vertices, this gives a way of transforming an axial map into a set of discrete elements suitable to be used as the input of an algorithm running on a computer. suitable for computational proposes. The axial graph does carry information about the connections of each line to all other lines of the map. However, all the information about the way the lines are distributed on the plane is lost.

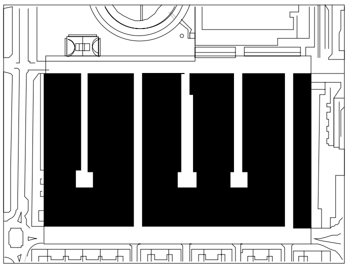


Figure 4. A segment of Alvalade, Lisbon.

Consider the axial map of Figure 5. It refers to a segment selected in Alvalade, a neighbourhood of Lisbon, which is represented in Figure 4. Alvalade was developed during the 1940's with the purpose of expanding the city. The plan was conceived to provide houses for about 45,000 inhabitants within different income groups. A concept of urban form based on cells was elaborated into a hierarchical principle. Housing groups are arranged as distinct areas (based on the primary school at the heart) and neighbourhood clusters (based on a major shopping and social centre sited on main roads).

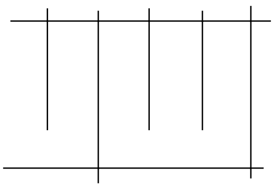


Figure 5. The corresponding axial map of figure 4.

Note that the axial map of Figure 5 and the one of Figure 2 share the same axial graph (Figure 3), despite having a different pattern of connections. Therefore, axial graphs do not characterize axial maps. To abbreviate this topological problem, the axial graph is complemented by a node map. This represents every intersection of lines in the axial map as a node and every axial line segment as a connection between nodes. Figures 6 and 7 show the node maps of the axial maps of Figures 2 and 5, respectively.

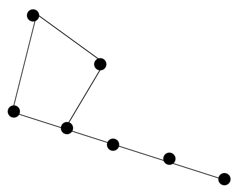


Figure 6. Node map of the axial map of figure 2.

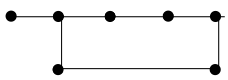


Figure 7. Node map of the axial map of figure 5.

Despite the pair (node map, axial graph) to be quite close to the axial map, characterizing the node map encloses the same problems as characterizing the axial map itself. In this paper we present a way of characterizing axial maps by means of an extension of a local property that enables the description of the cell decomposition of the plane induced by line arrangements. Line arrangements, that are particular cases of axial maps, consist of finite sets of lines laying on a plane in which every two lines intersect. These objects are studied within a mathematical theory known as *arrangements of (pseudo)lines* (see Chapter 6 of Björner et al (1993)). This gives a model to represent axial maps as sets of discret elements that makes configurational analysis efficiently possible. The proposed model is also much closer from axial maps than the usual representation by graphs.We hope, in this way, to give a satisfactory answer to the above question of Hillier et al (1993).

2 Graphs and axial maps

To make axial maps suitable for computation we must somehow find a class of *characterizing* objects which can be *encoded* or *represented*. An axial map is characterized by a given object if the object allows to recover all the information contained on the map. No two different maps correspond to the same object. The encoding or representation of an object consists of an appropriate description of the object as a sequence of symbols, in the sense that it can be used as the input of an algorithm runningon a computer (for details see Garey and Johnson (1979)).

Clearly, the deeper the knowledge of a certain class of objects is, the more interesting such class will reveal itself to be in handling axial maps. Especially, if that knowledge is based on results having algorithmic counterparts. The objects used in the literature to handle axial maps are two classes of graphs (Hillier and Hanson (1984),Krüger (1989), (1990)): the *node graph* and the *axial graph*.

The node graph of an axial map is a graph in which each vertice (v) corresponds to an intersection node (N_v). Vertices v and u are adjacent if and only if there is a line of the axial map containing N_v and N_u which does not intersect any other line along the segment connecting N_v with N_u. The node graph of the axial map of Figure 8 is represented in Figure 9. The above definition of node graph follows the traffic engineers' convention. In the context of space syntax, the definition also includes a labelling of the vertices indicating the number of lines that cross on the corresponding intersection nodes. This allows to recover all the lines of the axial map.We will use the traffic engineers' definition since the knowledge of how many lines cross on each intersecting node is irrelevant for the argument of this paper. We will use the traffic engineers' definition since the knowledge of how many lines cross on each intersectingnode is irrelevant in what follows. The axial graph of an axial map is a graph in which vertices correspond to lines. Two vertices are adjacent if and only if the corresponding lines of the axial map intersect. The axial graph of the map of Figure 8 is represented in Figure 10.

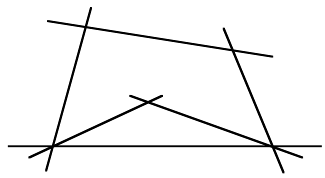


Figure 8. An axial map.

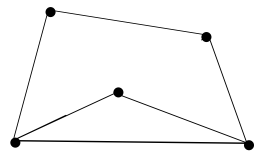


Figure 9. Node graph of figure 8.

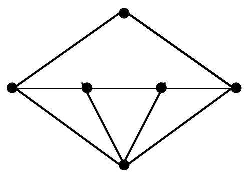
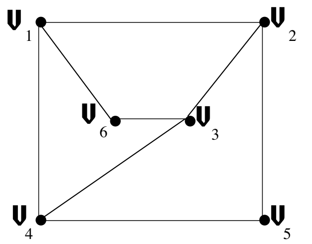


Figure 10. Axial graph of the axial map in figure 8.

There are two obvious advantages of using graphs in dealing with axial maps. First, graphs are easily encoded. Adjacency matrices, incidence matrices and adjacency lists (see, for example, Papadimitriou and Steiglitz (1982)) are some practical ways of encoding graphs. Second, graph theory furnishes an enormous amount of results and a quite good number of efficient algorithms for a wide variety of questions concerning graphs. Some of these questions are shown to be relevant in analyzing morphological properties behind axial maps. An example is the concept of *depth* of the vertices of a graph (Hillier and Hanson (1984)). The depth of vertice v is the sum of the distances from v to all other vertices. (The distance from v to vertice u is the number of edges of the shortest path connecting v with u). This measure on the vertices of the axial graph captures the notion of integration on the corresponding lines of the axial map.Vertices with low (high) depth values correspond to integrated (segregated) lines.

For a more curious reader, we point out that the list of depths is not an invariant under graph isomorphism. (If it were, the widely accepted conjecture stating that no polynomial time algorithm exists for deciding whether two graphs are isomorphic, would collapse!) Even for planar graphs (for which the graph isomorphism problem is solvable in polynomial time) we can find non-isomorphic graphs with the same list of depths. The planar graphs of Figure 11 both have depth values equal to 7, 7, 7, 7, 9 and 9 on vertices v_1, v_2, v_3 , v_4, v_5 and v_6, respectively.



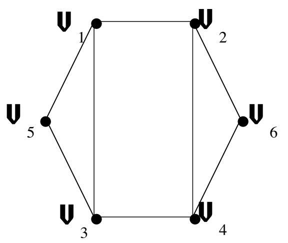


Figure 11. Non isomorphic planar graphs with the same list of depths.

Although graphs possess the above described good features, obviously neither node graphs, nor axial graphs do characterize axial maps. Also the pair (node graph, axial graph) is not a characterizing object. This follows from the fact that, in general, a planar graph has different immersions into the plane. As an example consider the axial map of Figure 12 which was obtained from a certain immersion into the plane of the node graph of Figure 9. This map has the same node and axial graphs as the map of Figure 8.

Among the different immersions of the node graph into the plane there is one that is specially interesting. This is called the *node map* (Krüger (1989), Hillier et al (1993)) which consists of an immersion compatible with the axial map. The node map is no more than the axial map itself where every line segment which does not have both ends coinciding with intersection nodes has been deleted. The node map of the axial map of Figure 8 is the planar representation used in Figure 9 to exhibit the node graph. The pair (node map, axial graph) is quite close to the axial map. In fact, except for the orientation of those lines (if any) which intersections occur on a unique point, the reconstruction of the axial map is possible. However, the difficulties of using this pair follow from the codification of the node map which encloses the same type of problems as the codification of the axial map itself.

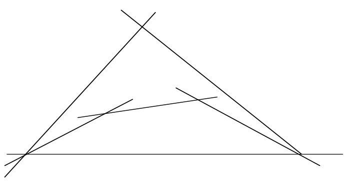


Figure 12. An axial map with the same node and axial graphs as the one of Figure 8

In the next section we introduce a class of objects which are suitable to be encoded, and that allow to recover the shape and the way the regions induced by axial maps are linked together. These objects consist of pairs of ordered sets derived from an orientation of the plane where the axial map lays. There are two good features about using these objects. First, the orders can be easily obtained and therefore encoding is efficiently achieved. Second, these objects nicely fit in the context of arrangement of lines. Since the pioneerwork of Grünbaum (Grünbaum (1970), (1972)) from the early seventies, this subject has grown as a fruitful mathematical theory, and a quite reasonable number of efficient algorithms have been developed (see Bokowski and Sturmfels (1989)). which will certainly reveal usefull if used in studing morphological properties described by axial maps.

3 Arrangements of lines

A *line arrangement* is a finite set of (affine) lines laying on a plane E, such that every two lines intersect. In what follows we will only consider arrangements consisting of at least three lines. We describe how to define certain orders on the set of lines and on the set of intersection nodes of a line arrangement. We use the arrangement of Figure 13 to illustrate the procedure.

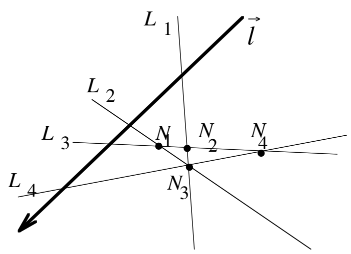


Figure 13. Defining orders for a line arrangement.

Add to the arrangement an oriented line ($\vec{l}$), such that:

(i) all the intersection nodes occur in one of the two half planes of $E \backslash \vec{l}$;

(ii) every line of the arrangement intersects $\vec{l}$;

(iii) the direction defined by any pair of intersection nodes is different from the direction of $\vec{l}$.

Note that (ii) and (iii) are always possible since we have infinite directions in the plane v_s a finite set offorbidden directions.

Let the m lines of the arrangement be labelled $L_1 < L_2 < ... < L_m$) by the order they meet

$\vec{l}$. Maintaining its direction, move the line $\vec{l}$ towards the half plane where the n intersection nodes occur and order them $N_1<N_2<...<N_n$ as they are met by $\vec{l}$. Note that this procedure can be implemented so to run efficiently. A line arrangement with a pair of orders of lines and intersection nodes as defined above will be called a *labelled arrangement*. Consider labelled arrangements consisting of three lines. One can easily realize that there are three possible types of such arrangements, namely type I, II and III, as represented by figures 14, 15 and 16.

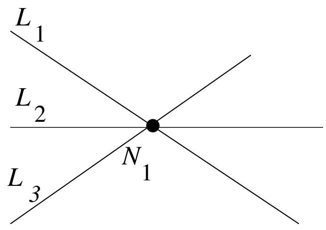


Figure 14. type I,
$N_1 = L_1 \cap L_2 \cap L_3$

33.5

Interestingly, it has been proved (see, for example, Theorem 6.6.4 in Björner et al (1993)) that every line arrangement can be fully described by identifying the types of all its three line subarrangements. More precisely, two arrangements, $A=(L, N)$, and $A'=(L', N')$ with m lines are *isomorphic* if and only if there are labels for A and A' such that, for every $i<j<k(\leq m)$, the labelled subarrangements $\{L_i,L_j,L_k\} \subseteq L$ and $\{L'_i,L'_j,L'_k\} \subseteq L'$ are of the same type.

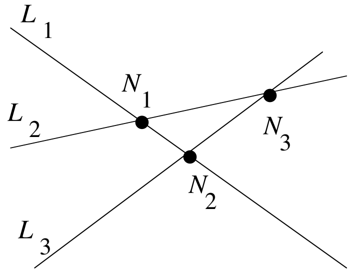


Figure 15. type II,
$N_1=L_1 \cap L_2<N_2=L_1 \cap L_3<N_3=L_2 \cap L_3$

4. Characterizing axial maps

We will now extend the above characterization of line arrangements to a convenient definition of isomorphic axial maps. We describe how to construct a line arrangement $A=(L, \bar{N})$ corresponding to any given axial map $AM=(L, N)$. First, consider the case in which no line in L will end at any point of intersection with another. Let us start with $\bar{N} := N$ and expand every line of the map. Whenever a new intersection node is created, add it to $\bar{N}$. Be aware, when expanding parallel lines, to disturb them slightly so that intersection will always be achieved. In this case, and as a consequence, the underlying line arrangement is not uniquely determined. See in Figure 18 a line arrangement corresponding to the axial map of Figure 17.

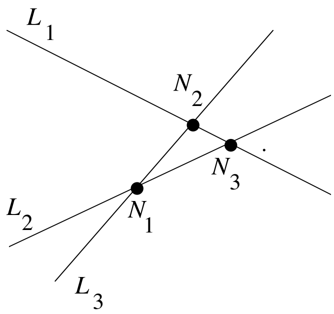


Figure 16. type III,
$N_1=L_2 \cap L_3<N_2=L_1 \cap L_3<N_3=L_1 \cap L_3$

Now use the procedure of section 3 to obtain the orderer lists $(L,<)$ and $(\bar{N},<)$. As we did for line arrangements, let us call an axial map with these induced orders on L and N a *labelled axial map*. For convenience, every configuration obtained by deleting any set of lines of an axial map will also be called an axial map, even if it is disconnected. There are seven different types of labelled axial maps with three lines. In addition to those of types I, II and III, we have types IV, V, VI, and VII represented in figures 19, 20, 21 and 22, respectively.

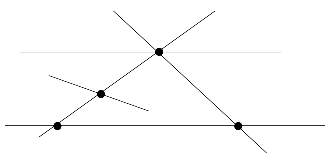


Figure 17. An axial map

As we have refered above, comparing line arrangements amounts to compare the corresponding three line subarrangements. By analogy we propose what seems to be a reasonable definition of isomorphic axial maps. We extend this property to what seems to be a reasonable definition of isomorphic axial maps. Two axial maps $AM=(L, N)$ and $AM'=(L', N')$ with m lines are *isomorphic* if there are labels for AM and AM' such that, for every $i<j<k$ ($\leq$ m), the labelled submaps $\{L_i, L_j, L_k\} \subseteq L$ and $\{L'_i,L'_j,L'_k\} \subseteq L'$ are of the same type.

We thus have a class of objects which are easily encoded, that characterize axial maps. These objects are: an ordered list of lines $(L,<) = (L_1, L_2, ..., L_m)$; an ordered list of intersection nodes $(N,<) = (N_1, N_2, ..., N_n)$ and, for $i = 1, 2, ..., n$, the identity N_i,

$$N_i = N_{i_1 i_2 ... i_s}, \quad (1)$$

describing the lines $L_{i_1} < L_{i_2} <...< L_{i_s}$ which intersect at node N_i.

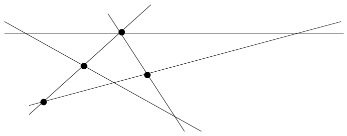


Figure 18. Line arrangement of map of figure 16.

33.6

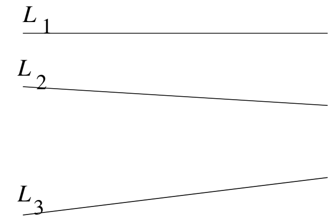


Figure 19. Axial map of type IV.

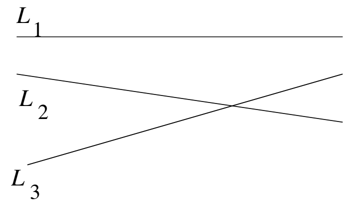


Figure 20. Axial map of type V.

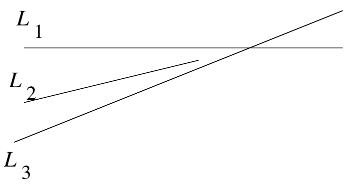


Figure 21. Axial map of type VI.

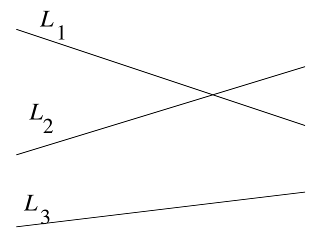


Figure 22. Axial map of type VII.

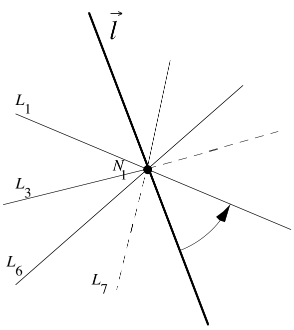


Figure 23. Defining the sequence of labels for node $N_i = N_{136\bar{7}1\bar{3}67}$

We now consider the case in which at least one line will end at an intersection point. We show how to adapt equality (1) in order to distinguish between lines that do and do not end at node N_i. Recall how the direction of $\vec{l}$ has been used to order the intersection nodes. Suppose $N_1 < N_2 < \ldots < N_{i-1}$ have already been defined and that $\vec{l}$ is now meeting the next node which is the intersection of lines $L_{i_1} < L_{i_2} < \ldots < L_{i_s}$. Instead of defining $N_i = N_{i_1 i_2 \ldots i_s}$, consider each line L_{i_j} as two half lines both ending at N_i and rotate $\vec{l}$ always to the same side, say counter-clockwise, until a complete turn is performed. When some half line L_{i_j} is met, let

$$l(ij) = \begin{cases} i_j & \text{if the half line } L_{ij} \text{ belongs to the axial map} \\ \bar{i}_j & \text{otherwise} \end{cases}$$

and define N_i as (see Figure 23)

$$N_i = N_{l(i_1)\ldots l(i_s) l(i_1)\ldots l(i_s)}. \qquad (2)$$

Two axial maps $AM=(L, N)$ and $AM'=(L', N')$ are *isomorphic* if, besides being isomorphic according to the former definition, the sequences (2) assigned to any pair of correspondingnodes in N and N' coincide.

We finish this section with the following remark. The proposed characterization of axial maps, as in the case of line arrangements, does not distinguish an axial map from any of its rotations. Moreover, which in some cases may turn out to be a more serious situation, it cannot distinguish an axial map from its mirror image. It may be that for some applications it would be desirable to consider the map as a more rigid configuration in order that only slight pertubations would not be perceptible. This can be achieved by adjusting the definition of isomorphic axial maps to an orientation of the plane given by a certain fixed referential. This amounts to fix once for all the oriented line $\vec{l}$ used to order the lines and intersection nodes. This way every map will be consider as a labelled one.

5. Final remark

In this paper we propose a characterization of axial maps which allows maps to be easily encoded and reconstruction efficiently achieved. We did not look into the use of this model in terms of syntax analysis. However, we believe it to be an adequate tool for a more specific definition of integration, leading to a more accurate quantification of the spatial distribution described by axial maps.

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Note

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